Jhargram Raj College

Problem Set - Special Functions

Code: Sem_3_Assignment_2

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-by S.S

1. Show that

   (a)
   $$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\Gamma(m)\Gamma(n)}{a^n(1+a)^m\Gamma(m+n)}$$

   (b)
   $$\int_0^1 \left(\ln\frac{1}{y}\right)^{n-1} dy = \Gamma(n)$$

   (c)
   $$\frac{1}{n}\int_0^\infty e^{-x^{1/n}}\,dx = \Gamma(n)$$

   (d)
   $$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\frac{\sqrt{\pi}}{n}$$

   (e) $\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5\cdot(2n-1)\sqrt{\pi}}{2^n}$

   (f)
   $$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

   (g) $2.5.8...(3n-1) = 3n\frac{\Gamma\left(n+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$

2. Prove that $\beta(p,q) = \beta(q,p)$

3. Show that
   $$\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx = \beta(p,q)$$

4. Show that
   $$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

5. Assuming $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ for $0 < n < 1$, show that
   $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$$

6. (a) Show clearly that
   $$\int_0^1 x^{m-1}(1-x^n)^{p-1}\,dx = \frac{1}{n}\beta\left(\frac{m}{n},p\right)$$
   where $n \neq 0$

   (b) Hence, find the exact value of
   $$\int_0^1 x^5(1-x^3)^2\,dx$$

7. Show that
   $$\int_0^1 \left(1-x^{1/3}\right)^{1/11} dx = \frac{1331}{1564}$$

8. Show that
   $$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n = \frac{2}{(n+1)(n+2)}$$

9. Show that
   $$\int_0^a \sqrt{x}\sqrt{a-x}\,dx = \frac{\pi a^2}{8}$$

10. Suppose $I_{m,n} = \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta$, where $m\in\mathbb{N}, n\in\mathbb{N}$

    (a) Show that $I_{m,n} = \frac{m-1}{m+n}I_{m-2,n}$

    (b) Hence, show further that
    $$\beta(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)}\beta(m-1,n-1)$$