বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. General Examinations 2020

(Under CBCS Pattern)

Semester - V

Subject: MATHEMATICS

Paper: DSE1AT

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.

The figures in the margin indicate full marks.

COMPLEX ANALYSIS

Answer any ***three*** from the following. 3×20=60

1. (a) Show that $\lim_{z\to 0}\frac{\overline{z}}{z}$ does not exist. 4

(b) Check the continuty and differentiability of the function $f(z)$ at the point $z=0$ where

$$f(z)=\begin{cases}\frac{z.\operatorname{Re}(z)}{|z|}, & z\neq 0\\ 0, & z=0\end{cases}$$ 8

(c) Show that a real function of a complex varibale either has derivative zero or the derivative does not exist. 4

(d) Find the residue of $f(z)=\dfrac{1}{\left(z^2+a^2\right)^2}$ at $z=ai$. 4

2. (a) Find the analytic function $f(z)=u+iv$ if $u-v=\dfrac{\cos x+\sin x-e^{-y}}{2(\cos x-\cosh y)}$ and $f\left(\dfrac{\pi}{2}\right)=0$. 8

(b) If u and v are harmonic functions conjugate to each other in some region G then show that u and v must be constant. 4

(c) Evalute : $\dfrac{1}{2\pi i}\int_C \dfrac{e^{zt}}{\left(z^2+1\right)^2}dz$ if $t>0$ and C is the circle $|z|=3$. 8

3. (a) Let C be the boundary of a square of vertices at the point $z=0,\ z=1,\ z=1+i,\ z=i$ taken in the counterclockwise sense. Show that $\int_C \pi e^{\pi\bar{z}}dz=4\left(e^{\pi}-1\right)$. 8

(b) Show that the function $f(z)$ defined by

$$f(z)=\begin{cases}\dfrac{\operatorname{Im}\left(z^2\right)}{\bar{z}}, & z\neq 0\\ 0, & z=0\end{cases}$$

satisfied the C-R equation at the origin yet it is not differentiable there. 8

(c) Expand $\cos z$ into Taylor series about the point $z=\dfrac{\pi}{2}$ and find the radius of convergence. 4

4. (a) Check the differentiability of the function $f(z)=e^x(\cos y-i\sin y)$. 4

(b) Using Cauchy residue theorem, evaluate $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)^2(z-2)}dz$ where C is the circle $|z|=3$. 6

(c) Evaluate : $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta$ 7

(d) If $f(z)=\frac{1+z}{1-z}$, determine $f'(2+3i)$. 3

5\. (a) Evaluate $\int_0^{\infty} \frac{x^2}{x^6+1}dx$ 10

(b) Determine a conjugate harmonic function *u*, where $u(x, y)=e^x \cos y$ in the complex plane *C*. 6

(c) Define extended complex plane. How do you represent geometrically on a sphere ? 4

6\. (a) Prove the necessary conditions that the function $f=u+iv$ is differentiable at a point $z_0=x_0+iy_0$ is that u_x, u_y, v_x, v_y exists and $u_x=v_y,\ v_x=-u_y$. 7

(b) State Liouville's Theorem. Derive the fundamental theorem of algebra using it. 7

(c) Prove or disporve: The point $z=0$ is the only ingularity of the function $f(z)=\sin\left(1-\frac{1}{z}\right)$ and $z=0$ is a simple pole. 6

MATRICES

Answer any ***three*** questions. 3×20=60

1. (a) Prove that the set of vectors $\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$ is linearly independent in $\mathbb{R}^3$.

   (b) Find the conditions on $a, b \in \mathbb{R}$, so that the vectors $\{(a, b, 1), (b, 1, a), (1, a, b)\}$ is linearly dependent.

   (c) For what values of k does the set $S = \{(k, 1, 1), (1, k, 1), (1, 1, k)\}$ form a basis for $\mathbb{R}^3$.

   (d) Prove that the set $S = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ is a basis for $\mathbb{R}^3$. 5+5+5+5

2. (a) Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ forms a subspae of $\mathbb{R}^3$. Find a basis and dimension of the subspace S of $\mathbb{R}^3$.

   (b) Find the inverse of the matrix A using elementary row operations on A.

   (c) Solve the system of equations $x + y + 3z = 0,\ 2x + y + z = 0,\ 3x + 2y + 4z = 0$.

   7+7+6

3. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the stanadard basis for $\mathbb{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_\alpha^\gamma$. 7+5

   (b) Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. 8

4. Let V be a vector space, and let $T: V \to V$ be linear. A linear subspace W of V is said to be T-invariant if $T(x) \in W$ for every $x \in W$, that is, $T(W) \subseteq W$. If W is T-invariant, we define the restriction of T on W to be the function $T_W : W \to W$ defined by $T_W(x) = T(x)$ for all $x \in W$.

   (a) Prove that the subspaces $\{0\}, V, R(T)$ and $N(T)$ are all T-invariant.

   (b) If W is T-invariant, prove that T_W is linear. 15+5

5. (a) Prove that the product of the eigen values of a square matrix A is det A.

   (b) If λ be an eigen value of a non-singular matrix A, then prove that λ^{-1} is an eigen values of A^{-1}.

   (c) Show that the eigen values of a real symmetric matrix are all real.

   (d) Find the eigen values and eigen vectors of the matrix $\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$. 5+5+5+5

6. (a) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

   (b) Determine the conditions for which the system of equations $x + y + z = 1$, $x + 2y - z = b$, $5x + 7y + az = b^2$ admits of (i) only one solution, (ii) no solution, (iii) many solutions. 10+10

LINEAR ALGEBRA

Answer any ***three*** questions. 3×20=60

1. (a) Prove that the intersection of two subspaces of a vector space V over a field F is a subspace of V. Is the union of two subspaces of V a subsapce of V ? Justify.

   (b) Prove that the set of vectors $\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$ is linearly independent in R^3.

   (c) Let $S = \{(x, y, z) \in R^3 : x + y + z = 0\}$. Prove that S is a subspace of R^3.

   (d) Show that $S = \{(x, y, z) \in R^3 : x + y - z = 0,\ 2x - y + z = 0\}$ is a subspace of R^3. Find the dimension of S. (4+3)+4+3+(3+3)

2. (a) Examine if the set S is a subspace of the vector space $R_{2\times 2}$, where

   (i) S is the set of all 2×2 real diagonal matrices;

   (ii) S is the set of all 2×2 real symmetric matrices.

   (b) Let V be a real vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{\alpha + \beta + \gamma,\ \beta + \gamma, \gamma\}$ is also a basis of V.

   (c) Define rank and nullity of a linear transformation.

   (d) Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear transformation. Prove that Kernel of T is a subspace of V. (3+3)+5+4+5

3. (a) Define a real vector space.

   (b) Define basis and dimension of a vector space V over a field F.

   (c) Define kernel and image of a linear mapping.

   (d) Let V and W be vector spaces over a field F. Let $T : V \to W$ be linear transformation and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V. Prove that the vectors

   $\{T(\alpha_1), T(\alpha_2), ..., T(\alpha_n)\}$ generate image of T.

(e) Define linear dependence and linear independence of a set of vectors. 5+4+4+4+3

4. (a) If $\alpha=(1,1,2)$, $\beta=(0,2,1)$, $\gamma=(2,2,4)$, determine whether α is a linear combination of β and γ.

(b) Let V be a vector spaces over a field F and let $\alpha, \beta \in V$. Then prove that the set $W=\{c\alpha+d\beta : c \in F,\ d \in F\}$ forms a subspace of V.

(c) Let $T: R^3 \to R^2$ be a linear transformation defined by

$T(x, y, z)=(3x-2y+z,\ x-3y-2z)$.

Find the matrix of T relative to the ordered bases

(i) $(1,0,0),(0,1,0),(0,0,1)$ of R^3 and $(1,0),(0,1)$ of R^2.

(ii) $(0,1,0),(1,0,0),(0,0,1)$ of R^3 and $(0,1),(1,0)$ of R^2.

(d) Let $T: R^3 \to R^3$ be a transformation defined by $T(x, y, z)=(x, y, 0)$. Prove that T is linear. 5+5+6+4

5. (a) Show that the set $S=\{(2,3,1,4),(3,2,4,1),(1,1,1,1)\}$ is linearly dependent in R^4.

(b) The matrix of a linear transformation $T: R^3 \to R^2$ relative to the ordered bases (0, 1, 1), (1, 0, 1), (1, 1, 0) of R^3 and (1, 0), (1, 1) of R^2 is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T.

(c) Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Then prove that T is injective if and only if $\ker(T)=\{\theta\}$.

(d) Define isomorphism of a linear transformation. 5+6+7+2

6. (a) Find $k \in R$ so that the set $S=\{(1,2,1),(k,3,1),(2,k,0)\}$ is linearly dependent in R^3.

(b) Define inverse of a linear transformation.

(c) Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Then prove that if T be linear then inverse of T is also linear.

(d) Find dimension of $S \cap T$, where S and T are subspaces of the vector space R^4 given by

$$S = \{(x, y, z, w) \in R^4 : 2x + y + 3z + w = 0\}$$

$$T = \{(x, y, z, w) \in R^4 : x + 2y + z + 3w = 0\}.$$

5+3+6+6

VECTOR CALCULUS AND ANALYTICAL GEOMETRY

Answer any ***three*** questions. 3×20=60

1. (a) If $\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$ then show that $\vec{a}$ and $\vec{b}$ are perpendicular. 4

   (b) If $\vec{a}$, $\vec{b}$ be two vectors such that $\left|\vec{a}\right|=10$, $\left|\vec{b}\right|=1$ and $\vec{a}.\vec{b}=6$, then find the value of $\left|\vec{a}\times\vec{b}\right|$. 4

   (c) Show that $\vec{\nabla}\left\{\frac{f(r)}{r}\vec{r}\right\}=\frac{1}{r^2}\frac{d}{dr}\left\{r^2 f(r)\right\}$, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r=\left|\vec{r}\right|$. 6

   (d) Show that the vector $\vec{F}=(2x-yz)\hat{i}+(2y-xz)\hat{j}+(2z-xy)\hat{k}$ is irrotational. Find a scalar point ϕ, such that $\vec{F}=\vec{\nabla}\phi$. 6

2. (a) Prove that $\vec{\nabla}.\left(\vec{\nabla}f\right)=\nabla^2 f$. 4

   (b) Find $\vec{\nabla}\phi$ where $\phi=r^n$, $r=\left|\vec{r}\right|$ and $\vec{r}=x\hat{i}+y\hat{i}+z\hat{i}$. 4

   (c) Reducing to canonical form, discuss the nature of the conic

   $4x^2-4xy+y^2+2x-26y+9=0$. 6

   (d) A plane passing through a fixed point (a,b,c) cuts the axes in *A*, *B* and *C*. Show that the locus of the center of the sphere *OABC* is $\frac{a}{x}+\frac{b}{y}+\frac{c}{x}=2$. 6

3. (a) Define divergence of a vector point function. 4

   (b) If *f* and *g* be two scalar point functions, then prove that $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f$. 4

   (c) Find the equation of the right circular cone which conains three positive coordinate axes. 6

(d) Find the equation of the right circular of radius 3 and whose axes is $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$. 6

4. (a) Find the nature of the conic $x^2+4xy+y^2-2x+2y+6=0$. 4

(b) Find the equation of the sphere having the center at (2, –3, 4) and radius equals to 5 unit. 4

(c) Find the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and which touches the plane $2x+2y-z=15$. 6

(d) Find the equation of the cone whose vertex is the point (1, 2, 3) and guiding curve is the circle $x^2+y^2+z^2=9,\ x+y+z=1$. 6

5. (a) Find the center and the radius of the sphere given by $2\left(x^2+y^2+z^2\right)-2x+4y-6z=15$. 4

(b) Find the equation of the sphere passing through the four points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c). 4

(c) Find the equation of the cylnder whose generators are parallel to the straight line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is $x^2+y^2=9,\ z=1$. 6

(d) Find the center and radius of the circle $x^2+y^2+z^2=25,\ x+2y+2z+9=0$. 6

6. (a) Find the eccentricity and foci of $\frac{x^2}{25}+\frac{y^2}{16}=1$. 4

(b) For the hyperbola $16x^2-9y^2=144$, find the vertices. 4

(c) Show that a necessary and sufficient condition for a scalar point function ϕ to be constant is that $\vec{\nabla}\phi=\vec{0}$. 6

(d) For every scalar point function ϕ, prove that curl of grad $\phi=0$. 6