বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - V

Subject: MATHEMATICS

Paper: C12T

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.
The figures in the margin indicate full marks.

Attempt any ***three*** questions. 3×20=60

1. (a) Let G be a group and $a, b \in G$. Then prove that $\theta_a \circ \theta_b = \theta_{ab}$ where θ_a, θ_b are the inner automorphisms cooresponding to *a*, *b*, respectively. 2

(b) Find all the characteristic subgroups of S_3. 2

(c) Prove that $Aut(\mathbb{Q}, +)$ is isomorphic with $(\mathbb{Q}^*, \bullet)$. 6

(d) (i) Let *G* be a group of order 2*m* where *m* is an odd integer. Show that *G* has a normal subgroup of order *m*. 5

(ii) Show that $|Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)| = 6$. 5

2. (a) Suppose G is a cyclic group. Then prove that every subgroup of G is a characteristic subgroup of G. 2

(b) Express the Klein's four group as an internal direct product of two of its proper subgroups. 2

(c) Let G' denote the commutator subgroup of a group G. Prove that G' is a normal subgroup of G and G/G' is abelian. 6

(d) Prove that $Aut(\mathbb{Z}_8)$ is isomorphic to the Klein's four group. 4

(e) Let G be a finite group and H be a subgroup of G such that $|H| = p^k$, where p is a prime and k is a non-negative integer. Then show that

$$[G:H] \equiv_p [N(H):H]$$

where $N(H) = \{g \in G : gHg^{-1} = H\}$ is the normalizer of the subgroup H in G. 6

3. (a) Prove that S_3 cannot be expressed as an internal direct product of its proper subgroups. 2

(b) Is the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$? Justify your answer. 2

(c) Find the comutator subgroup of S_3. Let H, K be subgroups of G such that $H \subseteq K$. If H is a characteristic subgroup of K and K is a normal subgroup of G then show that H is normal in G. 6

(d) Prove that for any group G, $|G/Z(G)| \neq 91$. 4

(e) Let G be a group of order pn where p is a prime and $p > n$. If H is a subgroup of order p of G then show that H is a normal subgroup of G. 6

4. (a) Let G be a group. Give an example of a group action of G on G. 3

(b) Let G be a group acting on a non-empty set S. Prove that the stabilizer of a, where $a \in S$, is a subgroup of G. 3

(c) Show that $I_3 = \{1, 2, 3\}$ is an S_3-set (i.e., the group S_3 acts on I_3) where the action is defined by $(\sigma, a) \to \sigma \,.\, a = \sigma(a)$ for all $\sigma \in S_3$ and for all $a \in I_3$. Find all the distinct orbits of S_3. Also find the stabilizers of 1, 2 and 3. 8

(d) Let G be a finite group and $a \in G$. Prove that $[G : C(a)] = 1$ if and only if $a \in Z(G)$ where $C(a)$ denotes the centralizer of a. 3

(e) State Sylow's first Theorem. 3

5. (a) Let H be a subgroup of order 11 and index 4 of a group G. Does G have a non-trivial proper normal subgroup ? Justify your answer. 2

(b) State class equation for a finite group. 2

(c) For any finite p-group G, p is prime, show that $Z(G) \neq \{e\}$. If G is a non-commutative group of order p^3 where p is a prime, show that $|Z(G)| = p$. 6

(d) State Fundamental Theorem for finite abelian groups. Describe all abelian groups of order 360 up to isomorphism. 2+4=6

(e) Let G be a finite p-group where p is a prime. Then using Cauchy's Theorem prove that $|G| = p^n$. 2

(f) Let G be a finite group having only two conjugacy classes. Show that $|G| = 2$. 2

6. (a) Consider the left action of the group $GL(2, \mathbb{R})$ on $GL(2, \mathbb{R})$ by conjugation. Find the stabilizer of $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. 2

(b) Show that every group of order 99 has a normal subgroup of order 9. 3

(c) For any finite p-group G, p is prime, show that $Z(G) \neq \{e\}$. If G is a non-commutative group of order p^3 where p is a prime, show that $|Z(G)| = p$. 3

(d) Let G be a group of order 273 = 3.7.13. Show that G has a cyclic subgroup of order 91. 6

(e) Using Sylow's theorems, prove that no group of order 56 is simple. 6

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