বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - III

Subject: MATHEAMATICS

Paper: C6T

(Group Theory - I)

Full Marks : 60

Time : 3 Hours

Candiates are required to give their answer in their own words as far as practicable.

The figures in the margin indicate full marks.

Answer any ***three*** from the following questions : 3×20

1. (a) (i) Let $(G, *)$ be a group and $a, b. \in G$. Suppose that $a^2 = e$ and $a * b^4 * a = b^7$. Then prove that $b^{33} = e$.

(ii) Consider $G = GL(2,\mathbb{R})$, where $GL(2,\mathbb{R})$ denotes the group of all invertible 2×2 real matrices and $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL(2,\mathbb{R}) \middle| ad > 0 \right\}$. Is H a subgroup of G ? Justify your answer.

(iii) Give and example of a group G and one of its subgroups H such that $aH = bH$ but $Ha \neq Hb$ for some $a, b \in G$

(iv) Show that there does not exist any isomorphism between the additive group of all real numbers $(\mathbb{R},\ +)$ and the multiplicative group of all non-zero real numbers $(\mathbb{R}^*,\ \cdot)$.

(v) If $(G,\ 0)$ be group and $a \in G$, prove that any conjugate of the element a has the same order as that of a. 2×5

(b) (i) Prove that a group $(G,\ *)$ is commutative if and only if $(a * b)^n = a^n * b^n$ for any three consecutive integers n and for all $a, b \in G$. 5

(ii) (a) Show that $H = \{e, (1\ \ 2)(3\ \ 4)\}$ is not a normal subgroup of S_4.

(b) Find all homomorphisms of the group $(\mathbb{Z},\ +)$ to itself. 2+3=5

2. (a) (i) Give example of two subgroups H, K such that their product HK is not a subgroup.

(ii) Let G be a group and S be a non-empty subset of G. Define normalizer of S in G. Then show that the normalizer of S is a subgroup of G.

(iii) Suppose N is a normal subgroup of a group G. If G is an abelian group then G/N is a cyclic group — is this statement true ? Give logic in support of your answer.

(v) Show that $GL(2,\ \mathbb{R})/SL(2,\ \mathbb{R})$ is isomorphic with $(\mathbb{R}^*,\ \cdot)$. 2+(1+2)+2+3=10

(b) (i) Let S be a non-empty subset of a group G. If $< S >$ denotes the subgroup of G generated by S, then prove that

$$< S > = \left\{ \prod_{i=1}^{n} s_i^{e_i} \mid s_i \in S,\ e_i = \pm 1, i = 1, 2, ..., n;\ n \in \mathbb{N} \right\}.$$

(ii) Using First Isomorphism theorem show that $\mathbb{Z}_9$ is not a homomorphic image of $\mathbb{Z}_{16}$.

(iii) Prove that $K = \{e, (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}$ is a normal subgroup of A_4. 5+2+3=10

3. (a) (i) Prove that $H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2,\ \mathbb{R}) \,\middle|\, \text{either } a \neq 0 \text{ or } b \neq 0 \right\}$ is a subgroup of $GL(2,\ \mathbb{R})$.

(ii) Prove that any finitely generated subgroup of $(\mathbb{Q},\ +)$ is cyclic.

(iii) Prove that every group of order 49 contains a subgroup of order 7.

(iv) Let G be a group. Prove that the mapping $f : G \to G$ defined by $f(a) = a^{-1}$ is a homomorphism if and only if G is commutative. 3+3+2+2=10

(b) (i) Let G be a group such that $|G| < 320$. Suppose H, K be two subgroups of G such that $|H| = 35$, $|K| = 45$. The find the order of G.

(ii) Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

(iii) Let K be a normal subgroup of a group G such that $[G:K] = m$. If n is a positive integer such that $\gcd(m,\ n) = 1$, then show that $K \supseteq \{g \in G \mid o(g) = n\}$.

(iv) Prove that up to isomorphism there are only two groups of order 4.
2+2+2+4=10

4. (a) (i) Prove that every group of order p^2 is abelian, where p is prime.

(ii) Let G be cyclic group and H be a subgroup of G. Prove that factor group $\frac{G}{H}$ is cyclic. Is the converse true ? Justify. 4+6

(b) (i) State the converse of Lagrange's theorem for finite group and justify with an example whether the converse of Lagrange's theorem is true or false.

(ii) Show that the (external) direct product $\mathbb{Z}_3 \times \mathbb{Z}_4$ of the cyclic group $\mathbb{Z}_3$ and $\mathbb{Z}_4$ is a clyclic group. 1+5+4

5. (a) Two finite cyclic groups of the same order are isomorphic. 10

(b) (i) Let G be a group of order 10 and G' be a group of order 6. Prove that there does not exists a homomorphism of G onto G'.

(ii) Let $G = (\mathbb{Z}_5, +)$ and $\varphi : G \to G$ be defined by $\varphi(\overline{x}) = 3\overline{x}$, $\overline{x} \in \mathbb{Z}_5$. Find $\ker \varphi$. 10

6. (a) (i) Prove that every proper subgroup of a symmetric group S_3 is cyclic.

(ii) Let H be a subgroup of a group G and, $b \in G$. Prove that $aH \cap bH = \varphi$ if and only if $b \notin aH$.

(iii) Let P and Q be subgroups of a group G and $\gcd(0(P), 0(Q)) = 1$. Prove that $P \cap Q = \{e_G\}$. 4+3+3

(b) (i) Prove that the order of every subgroup of a finite group G is a divisor of the order of G.

(ii) Prove that a non-commutative group of a order $2n$, where n is odd prime, must have a subgroup of order n.

(iii) Prove that a group of order 27 must have a subgroup of order 3. 5+3+2