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UG/5th Sem/Math(H)/T/19

2019

B.Sc. (Honours)

5th Semester Examination

MATHEMATICS

Paper - DSE-1T

Full Marks : 60

Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

(LINEAR PROGRAMMING)

Unit - I

(Simplex Algorithm)

1. Answer any *five* from the following : 2×5

(a) Find a basic feasible solution of the system of equations :

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 + x_2 + 4x_3 = 4$$

Is the solution degenerate ?

[Turn Over]

(b) Prove that a hyperplane is a convex set.

(c) Prove that the set of all convex combination of a finite number of points is a convex set.

(d) What is simplex ? Give an example of a simplex in 3-dimension.

(e) What is the basic principle of two phase method?

(f) Define convex polyhedron.

(g) Put the following LPP in a standard form

Minimize $Z = 3x_1 - 4x_2 - x_3$

Subject to,

$$\begin{aligned} x_1 + 3x_2 - 4x_3 &\le 12 \\ 2x_1 - x_2 + x_3 &\le 20 \\ x_1 - 4x_2 - 5x_3 &\ge 5 \end{aligned}$$

$x_1 \ge 0$, x_2 and x_3 are unrestricted in sign.

(h) (i) State Fundamental theorem of L.P.P.

(ii) State the sufficient condition for a basic feasible solution X_B to an L.P.P.

Maximize $Z = C^T X$

Subject to, $AX = b, \ X \geq 0$

to be optimal.

2. Answer any *one* from the following : 5×1

(a) Find the solution of the following L.P.P. graphically

Maximize $Z = 5x_1 + 3x_2$

Subject to $3x_1 + 5x_2 \leq 15$; $5x_1 + 2x_2 \leq 10$.

(b) A firm manufactures three products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4. respectively for each unit of products.

The firm has two machines and below is the required processing time in minutes for each machine on each product.

Machines X and Y have 2000 and 2500 machine-minutes respectively.

Product

		A	B	C
Machines	X	4	3	5
	Y	2	2	4

The firm manufactures 100 A's, 200 B's and 50 C's but not more than 150 A's.

Set up a L.P.P. to maximize the profit.

3. Answer any *one* from the following : 10×1

(a) Solve using two phase method :

Minimize $Z = 3x_1 + 5x_2$

Subject to
$$x_1 + 2x_2 \geq 8$$
$$3x_1 + 2x_2 \geq 12$$
$$5x_1 + 6x_2 \leq 60, \quad x_1, x_2 \geq 0.$$

(b) Solve by simplex method (penalty method).

Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to
$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Unit - II

(Duality and Special LPP)

4. Answer any *three* from the following : 2×3

(a) State the weak duality theorem of a L.P.P.

(b) Give the comparison between transportation and assignment problem.

(c) Find the dual of the following LPP :

Maximize $Z = 6x_1 + 5x_2 + 10x_3$

$$\begin{aligned} \text{Subject to}\quad 4x_1 + 5x_2 + 7x_3 &\le 5 \\ 3x_1 + \quad\;\; + 7x_3 &\le 10 \\ 2x_1 + x_2 + 8x_3 &= 20 \\ 2x_2 + 9x_3 &\ge 5 \end{aligned}$$

$x_1, x_3 \ge 0$ and x_2 is unrestricted in sign.

(d) Prove that dual of dual is primal.

(e) Prove that the number of basic variables in a transportation problem is at most $(m+n-1)$.

5. Answer any *one* from the following : 5×1

(a) If x be any feasible solution to the primal problem and v be any feasible solution to the dual problem, then $cx \leq b'v$. 5

(b) Solve the travelling salesman problem where the entries as given as distance. Find minimum distance.

	A	B	C	D	E
A	–	7	6	8	4
B	7	–	8	5	6
C	6	8	–	9	7
D	8	5	9	–	8
E	4	6	7	8	–

6. Answer any *one* from the following : 10×1

(a) Find the dual of the following problem and solve the dual problem. Also find the solution of the primal problem from the dual.

Maximize $Z = 6x_1 + 4x_2 + 6x_3 + x_4$

Subject to $4x_1 + 5x_2 + 4x_3 + 8x_4 = 21$

$$3x_1 + 7x_2 + 8x_3 + 2x_4 \leq 48$$

$$x_1, x_2, x_3, x_4 \geq 0$$

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(b) (i) Find the optimal solution of the transportation problem

	D_1	D_2	D_3	D_4	
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
	20	40	30	10	

using VAM method. 6

(ii) Find the optimal assignments for the assignment problem with the following cost matrix

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

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Unit - III

(Game Theory)

7. Answer any *two* from the following : 2×2

(a) State the general dominance rules to reduced the size of pay-off matrix.

(b) Solve the game with the following payoff matrix

		B_1	B_2
A	A_1	1	3
	A_2	4	2

(c) State the following terms in concern with the game theory :

Rectangular game, Saddle point, Symmetric game.

8. Answer any *two* from the following : 5×2

(a) Solve the Game graphically :

	B_1	B_2	B_3	B_4
A_1	−1	3	2	2
A_2	6	2	5	3

(b) Transform to LPP and solve the game problem whose payoff matrix is given below, by simplex method.

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

(c) Use dominance property to reduce the payoff matrix and solve the game

0	0	0	0	0	0
4	2	0	2	1	1
4	3	1	3	2	2
4	3	7	−5	1	2
4	3	4	−1	2	2
4	3	3	−2	2	2

(POINT SET TOPOLOGY)

Unit - I

(Countable and Uncountable sets)

(Mark - 18)

1. Answer any *four* from the following : $2\times4=8$

   (a) Prove that $\mathbb{N}\times\mathbb{N}$ is countable where $\mathbb{N}$ denotes the set of all positive integers. 2

   (b) State Schröder-Bernstein Theorem. 2

   (c) State Axiom of choice. 2

   (d) Is the set of all integers well-ordered ? Justify your answer. 2

   (e) Define strict partial order on a non-empty set. 2

   (f) State the Zorn's lemma. 2

2. Answer any *two* from the following : $5\times2=10$

   (a) Prove that there is no surjective map $\mathbb{N}$ to the power set of $\mathbb{N}$ where $\mathbb{N}$ denotes the set of all positive integers. 5

(b) (i) Let A be a set. Define the Cardinal number of A which is often denoted by Card (A). 2

(ii) Let A, B be two disjoint sets. If we define the addition of cardinal nos. of A & B as follows :

$$\text{Card}(A) + \text{Card}(B) = \text{Card}(A \cup B),$$

then check whether the above definition is well-defined or not. 3

(c) If a and b are two real numbers, define $a < b$ if and only if $b - a$ is a positive rational number. Show that '<' is a strict partial order on $\mathbb{R}$. Then exhibit one maximal simply ordered proper subset of $\mathbb{R}$ (with justification). 2+3=5

Unit - II

(Topological Space)

(Mark - 21)

3. Answer any *three* from the following : 3×2=6

(a) Define lower limit topology on $\mathbb{R}$. 2

(b) Let X, Y, Z be three topological spaces and $f: X \to Y$, $g: Y \to Z$ be both continuous. Then prove that $g \circ f$ is a continuous map from X to Z. 2

(c) Let τ_1, τ_2 be two topologies on a non-empty set X. Suppose $\tau_1 \subset \tau_2$. Let A be a non-empty subset of X. Then show that every T_2-limit point of A is a τ_1-limit point of A, too. 2

(d) Let (X, τ_1) and (Y, τ_2) be two topological spaces. Define product topology of X and Y. 2

(e) Let (X, d) be a metric space. The exhibit a basis $\mathscr{B}$ for the metric topology induced by d on X. 2

(f) State Baire Category Theorem. 2

4. Answer any *one* from the following : 5×1=5

(a) Let (X, τ) and $(Y, \mathscr{U})$ be two topological spaces and $f: X \to Y$ be a mapping. Then prove that the following are equivalent.

(i) f is continuous.

(ii) $f^{-1}(S) \in \tau$ for all $S \in \mathscr{S}$ where $\mathscr{S}$ is a subbase for $\mathscr{U}$. 5

(b) Let (X, τ) and $(Y, \mathscr{U})$ be two topological spaces. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases corresponding to τ and $\mathscr{U}$, respectively. Then prove that

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B},\ C \in \mathscr{B}'\}$$

forms a basis for the product topology on $X \times Y$. 5

5. Answer *one* from the following : $10 \times 1 = 10$

(a) (i) Let $X = \{a, b, c, d, e\}$. Verify whether each of the following collections of subsets of X forms a topology on X (Give reasons).

(1) $\tau_1 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

(2) $\tau_2 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

(3) $\tau_3 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$.

2+2+2=6

(ii) Define finite complement topology on a non-empty set X and prove that if X is finite then the finite complement topology on X is same as the discrete topology on X.

2+2=4

(b) (i) Let $X = A \bigcup B$ where A, B are closed in (X, τ). Let $f : A \to Y$ and $g : B \to Y$ be two continuous functions. If $f(x) = g(x)$ for all $x \in A \bigcap B$, then prove that f and g combine to give a continuous function $h : X \to Y$ defined by

$$h(x) = f(x), \text{ when } x \in A$$
$$= g(x), \text{ when } x \in B.$$

5

(ii) Let (X, d) be a metric space. Let us consider another metric $\bar{d}$ on X defined by

$\bar{d}(x, y) = \min\{1, d(x, y)\}$ for all

$x, y \in X$.

Then prove that the metric topology on X induced by d is same as the metric topology on X induced by $\bar{d}$.

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Unit - III

(Connectednes and Compactness)

(Mark - 21)

6. Answer any *three* from the following : $2 \times 3 = 6$

(a) Define a connected topological space. 2

(b) Define path component of a topological space. 2

(c) Consider $(\mathbb{R}, \tau_u)$ where τ_u denotes the usual topology on $\mathbb{R}$. Is $(\mathbb{R}, \tau_u)$ compact ? Justify your answer. 2

(d) Give example (with justification) of a locally compact space which is not compact. 2

(e) Consider $\mathbb{R}^n$ with the Euclidean metric *d*. In the metric topology on $\mathbb{R}^n$ induced by *d*, find the compact subspaces of $\mathbb{R}^n$. 2

(f) "Union of connected subspaces of any topological space is connected" — true or false ? Justify your answer. 2

7. Answer any *one* from the following : $5\times1=5$

(a) Let (X, τ) be a topological space and $\{A_\alpha\}_{\alpha\in J}$ be a family of connected subspaces of X. Prove that if $\bigcap_{\alpha\in J} A_\alpha \neq \phi$ then $\bigcup_{\alpha\in J} A_\alpha$ is connected. 5

(b) Prove that every closed subspace of a compact space is compact. 5

8. Answer any *one* from the following : $10\times1=10$

(a) (i) Prove that product of two connected spaces is connected. 5

(ii) Show that continuous image of a compact space is compact. 5

(b) (i) Define totally bounded metric space.

(ii) Prove that if (X, d) is a totally bounded metric space then d is a bounded metric on X.

(iii) Prove that $(\mathbb{R}, \overline{d})$ is bounded but not totally bounded where $\overline{d}$ is a metric defined on $\mathbb{R}$ by

$$\overline{d}(x, y) = \min\{1, |x-y|\}.$$

(iv) Give example of a metric subspace Y of $(\mathbb{R}, d)$ (where $d(a, b) = |a-b|$ for all $a, b \in \mathbb{R}$) such that Y is totally bounded but not complete. 2+3+3+2=10

(THEORY OF EQUATIONS)
Unit - I

(Properties of Polynomial Equation)

(Marks : 15)

1. Answer any *five* questions : $2 \times 5 = 10$

(a) If α be the imaginary root of the equation $x^n - 1 = 0$, where n is prime, prove that

$$(1-\alpha)(1-\alpha^2)\ldots(1-\alpha^{n-1}) = n.$$

(b) Express $f(x) = x^4 - 2x^3 - 5x^2 + 10x - 3$ in the form $\left(x^2 - x + \lambda\right)^2 - (ax + b)^2$.

(c) If $x^2 + px + 1$ be a factor of $ax^3 + bx + c$ prove that $a^2 - c^2 = ab$.

(d) Apply Descartes rule of signs to find the nature of the roots of the equation $x^8 + 1 = 0$.

(e) If α, β, γ are the roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then find the centroid of the triangle having vertices

$$\left(\alpha, \frac{1}{\alpha}\right), \left(\beta, \frac{1}{\beta}\right), \left(\gamma, \frac{1}{\gamma}\right).$$

(f) If $f(x)$ be a polynomial of degree n then prove that $f(x) = 0$ cannot have more than n roots.

(g) If 1, a, b, ..., k are n roots of $x^n - 1 = 0$ then prove that $(1-a)(1-b)...(1-k) = n$.

(h) How many times the graph of the polynomial $\left(x^4-4\right)\left(x^2+x+2\right)$ will cross x-axis ?

2. Answer any *one* question : $5\times1=5$

(a) If the equation $x^4+px^3+qx^2+rx+s=0$ has roots of the form $\alpha\pm i\alpha$, $\beta\pm i\beta$, where α, β are real, prove that $p^2-2q=0$ and $r^2-2qs=0$. Hence solve the equation
$x^4+6x^3+18x^2+24x+16=0$.

(b) If the equation $f(x)=0$ has all its roots real, then show that the equation $ff''-(f')^2=0$ has all its roots imaginary.

Unit - II

(Symmetric Function I)

(Marks : 16)

3. Answer any *three* questions : $2\times3=6$

(a) If α be a special root of the equation $x^8-1=0$, then prove that

$$(\alpha+2)(\alpha^2+2)...(\alpha^7+2)=\frac{2^8-1}{3}.$$

(b) If α, β, γ be the roots of the equation $x^3+qx+r=0$, then find the value of $\sum\frac{1}{\alpha^2-\beta\gamma}$.

(c) If the roots of the equation $x^3+ax^2+bx+c=0$ are in GP. then prove that $b^3=a^3c$.

(d) If α, β, γ be the roots of the equation $x^3+ax+b=0$, find $\sum\alpha^5$.

(e) Show that all imaginary roots of $x^7=1$ are special roots.

4. Answer any *one* question : $10\times1=10$

(a) (i) Solve

$x^3-12x+8=0$ by Cardan's method.

(ii) Find the special roots of the equation $x^9-1=0$. Deduce that

$2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$, $2\cos\frac{8\pi}{9}$ are the roots of the equation $x^3-3x+1=0$. 4+(2+4)

(b) (i) Reduce the reciprocal equation

$3x^6+x^5-27x^4+27x^2-x-3=0$

to a reciprocal equation in the standard form and solve it. 5

(ii) If α, β, γ be the roots of the equation

$x^3+px^2+qx+r=0 \, (r \neq 0)$,

find the equation whose roots are

$\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}$, $\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\alpha}$, $\frac{1}{\alpha}+\frac{1}{\gamma}-\frac{1}{\beta}$ 5

Unit - III

(Symmetric Function II)

(Marks : 14)

5. Answer any *two* questions : 2×2=4

(a) Prove that the roots of the equation

$$(2x+3)(2x+4)(x-1)(4x-7)+(x+1)(2x-1)(2x-3)=0$$

are all real and different. Separate the intervals in which the roots lie.

(b) Obtain the equation whose roots are the square of the roots of the equation

$$x^4 - x^3 + 2x^2 - x + 1 = 0.$$

(c) The sum of two roots of the equation

$x^3 + a_1x^2 + a_2x + a_3 = 0$ is zero, show that $a_1a_2 - a_3 = 0$.

6. Answer any *two* questions : 5×2=10

(a) Let

$$f(x) \equiv x^n + p_1x^{n-1} + ... + p_{n-2}x^2 + p_{n-1}x + p_n = 0$$

be an equation of degree n having roots $\alpha_1, \alpha_2, ..., \alpha_n$. Let $s_r = \alpha_1^r + \alpha_2^r + ... + a_n^r$, where $r \geq 0$ is an integer. Then prove that

$$s_r + p_1s_{r-1} + ... + p_{r-2}s_2 + p_{r-1}s_1 + rp_r = 0,$$

if $1 \leq r < n$.

(b) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, $a \neq 0$ then prove that

$$f'(\alpha)+f'(\beta)+f'(\gamma)+f'(\delta)=\frac{32}{a^2}\left[3abc-a^2d-2b^3\right].$$

(c) If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4+px^3+qx^2+rx+s=0$, such that

$\alpha\beta+\gamma\delta=0$, then prove that

$p^2s+r^2-4qs=0$.

Unit - IV

(Strum's Theorem)

(Marks : 15)

7. Answer any *one* question : 5×1=5

(a) Reduce the cubic equation

$ax^3+3bx^2+3cx+d=0$ (a, b, c, d are real)

to the standard form $Z^3+3HZ+G=0$ where G and H are function of a, b, c, d. Hence obtain necessary and sufficient condition in terms of G and H for the cubic to have two equal roots.

(b) Find the number of the real roots of the equation $x^4+4x^3-x^2-2x-5=0$ by using Sturm's method.

[Turn Over]

8. Answer any *one* question : $10\times1=10$

(a) Define Sturm's functions. Find the Sturm's functions of the polynomial

$f(x)=x^5-5ax+4b$. If a and b are positive, prove that the equation $x^5-5ax+4b=0$ has three real roots or only one real root according as $a^5 > or < b^4$. 2+3+5

(b) (i) Find the transformation $x=\lambda y+\mu$ which will change the equation

$x^4+4x^3-18x^2-44x-7=0$

into reciprocal form. Hence solve the equation.

(ii) If α be a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$, show that

$\alpha=-\dfrac{8d}{3c}$. 7+3
