Total Pages - 6 UG/3rd Sem/MATH(H)/19

2019

B.Sc.

3rd Semester Examination

MATHEMATICS (Honours)

Paper - C 6-T

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Group Theory - I

Unit - I

1. Answer any *two* questions : $2\times2=4$

(a) Define Dihedral group.

(b) Let G be a group and $a \in G$, $0(a) = 12$. Find $0(a^3)$ and $0(a^8)$.

(c) Let $(G, \circ)$ be a group and $a, b \in G$. If $a^2 = e$ and $a \circ b^2 \circ a = b^3$, prove that $b^5 = e$.

2. Answer any *one* question : 5×1=5

(a) Let $G=\left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in Q^* \right\}$,

where $Q^* = Q - \{0\}$. Then prove that G is an abelian group with respect to multiplication of matrices.

(b) Let (G, o) be a semigroup and for any two elements a, b in G, each of the equations $a \circ x = b$ and $y \circ a = b$ has a solution in G. Prove that (G, o) is a group.

Unit - II

3. Answer any *two* questions : 2×2=4

(a) Let G be a group. Show that the centre of the group G is a subgroup of G.

(b) Prove that centralizer of an element in a group G is a subgroup of G.

(c) Show by an example that a non abelian group can have an abelian subgroup.

4. Answer any *two* questions : $2\times5=10$

(a) Let H and K are subgroups of a group G such that HK = $\{hk : h\in H \text{ and } k\in K\}$ is a subgroup of G. Then prove that

$$0(HK)=\frac{0(H)0(k)}{0(H\cap K)}.$$

(b) Define centre of a group G. Find centre of S_3.

(c) Let $(G,\circ)$ be an abelian group and n be a fixed positive integer. Let $H\left\{a^n : a\in G\right\}$. Prove that $(H,\circ)$ is a subgroup of $(G,\circ)$.

Unit - III

5. Answer any *two* questions : $2\times2=4$

(a) Find all orders of subgroups of the group Z_{10}.

(b) Find all left cosets of $H=\{\bar{0},\bar{3}\}$ in the group $G=(Z_6\ +)$.

(c) If $S=\{1,\alpha,\alpha^2,\dots\alpha^{11}\}$ form a cyclic group generated by α under multiplication then find

$\langle a^4 \rangle$

$o(A)$ where $A = \langle \alpha^4 \rangle$ is a subgroup of $(S, \circ)$.

6. Answer any *one* question : $10\times1=10$

(a) (i) If G be a cyclic group of prime order p, prove that every non-identity element of G is a generator of the group. 5

(ii) Prove that the order of a permutation on a finite set is the l.c.m. of the lengths of its disjoint cycles. 5

(b) (i) Prove that in a finite group G, order of any subgroup divides order of the group G. Does the converse true ? Justify your answer with example. 5+1

$o(t)$

(ii) Let $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z} \right\}$. Prove that G is a cyclic group with respect to the usual multiplication of matrices. 4

Unit - IV

7. Answer any *two* questions : $2\times2=4$

(a) Prove that if H has index 2 in G, then H is normal in G

(b) Write down all the elements of the factor group G/H and also Cayley table :

$G = Z_6$ and $H = \{\bar{0}, \bar{3}\}$.

(c) Show that alternating group of symmetric group of degree three is normal subgroup.

8. Answer any *one* question : $10\times1=10$

(a) (i) Let G be a cyclic group of order 12 generated by a and H be the cyclic subgroup of G generated by a^4. Prove that H is normal in G. Verify that the quotient group $\frac{G}{H}$ is a cyclic group of order 4. 6

(ii) Prove that enery group of order p^2 is abelian, where pio a prime. 4

(b) (i) Find the number of elements of order 5 in $Z_{15} \times Z_5$. 5

(ii) Let G_1 and G_2 be two groups and $G = G_1 \times G_2$ be the direct product of G_1 and G_2. Prove that $H_1 = \{(g_1, e_2) \mid g_1 \in G_1, e_2 = \text{identity of } G_1\}$ and normal

$$H_2 = \{(e_1, g_2) \mid g_2 \in G_2, e_1 = \textit{identity of } G_1\}$$

are normal subgroups of G

Unit - V

9. Answer any *two* questions : $2\times2=4$

(a) If $\phi : G \to G'$ be a group homomorphism, prove that $\phi(e) = e'$ and $\phi(x^{-1}) = \phi(x)^{-1}$. $\forall x \in G$.

(b) Let $G = S_3$, $G' = (\{1, -1\}, \cdot)$ and a mapping $\phi : G \to G'$ be defined by

$\phi(\alpha)$ $\alpha\cdot\beta =$ $\phi(\alpha\beta) = -1$

$$\phi(\alpha) = \begin{cases} -1, & \text{if } \alpha \text{ is even permutation in } S_3 \\ 1, & \text{if } \alpha \text{ is odd permutation in } S_3 \end{cases}$$

Examine if ϕ is a homomorphism.

(c) Show that the groups (Q, +) and (R, +) are not isomorphic.

10. Answer any *one* question : $1\times5=5$

(a) State and prove first isomorphism theorem on groups. $1+4$

(b) Find all homomorphisms from the group $(Z_8, +)$ to $(Z_6, +)$. 5