2019

B.Sc. (Honours)

4th Semester Examination

MATHEMATICS

Paper - C8T

(Riemann Integration and Series and functions)

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Unit - I

(Riemann Integration)

[Marks 19]

1. Answer any ***two*** questions 2×2

(a) A function f is defined on [1, 3] by $f(x) = [x^2]$.

Evaluate $\int_1^3 f(x)\,dx$. 2

[Turn Over]

(b) If a function $f : [a, b] \to R$ be integrable on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$, then prove that $\int_a^b f \geq 0$. 2

(c) If f be defined on $[-2, 2]$ by

$$f(x) = 3x^2 \cos\frac{\pi}{x^2} + 2\pi \sin\frac{\pi}{x^2}, \quad x \neq 0$$
$$= 0, \quad x = 0,$$

then show that f is integrable on $[-2, 2]$.

Evaluate $\int_{-2}^{2} f$. 1+1

2. Answer any ***one*** question : 5×1

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_{x-c}^{x+c} f(t)dt$. Show that $g(x)$ is differentiable on $\mathbb{R}$ and find $g'(x)$. 4+1

(b) State Bonnet's form of second mean value theorem of integral calculus. Hence establish

$$\left|\int_a^b \sin x^2\right| \le \frac{1}{a} \text{ in } 0 < a < b < \infty.$$ 2+3

3. Answer any ***one*** question : 10×1

(a) (i) State and prove Darboux theorem. 5

(ii) If a function $f : [a, b] \to R$ be integrable on $[a, b]$ then prove that the function F defined by $F(x) = \int_a^x f(t)dt$, $x \in [a, b]$ is differentiable at any point $c \in [a, b]$ at which f is continuous and $F'(c) = f(c)$. 5

(b) (i) If a function $f : [a, b] \to R$ be integrable on $[a, b]$ then prove that $|f|$ is integrable on $[a, b]$. Is the converse true? 4+1

(ii) Define Riemann sum for a function f. A function f is defined on $[0, 1]$ by

$f(x) = 1$, if x is rational

$= 0$, if x is irrational.

[Turn Over]

Using Riemann sums, show that f is not integrable on [0, 1]. 1+4

Unit - II

[Improper Integrals]

[Marks 11]

4. Answer any ***three*** questions : 2×3

(a) Prove that $\Gamma(n+1) = n\Gamma(n), n > 0$. 2

(b) Using μ test, show that $\int_1^\infty \frac{1}{x(1+x^2)} dx$ is convergent. 2

(c) Using comparison test, show that $\int_0^1 \frac{x^{p-1}}{1+x} dx$ is convergent if $p > 0$ and is divergent if $p \le 0$. 2

(d) State Dirichlet test for the convergence of an improper integral. 2

(e) Show that $\int_0^{\pi/2} \frac{x^m}{\sin^n x} dx$ is convergent iff $n < 1 + m$. 2

5. Answer any ***one*** question : 5×1

Examine the convergence of the integrable

(i) $\int_0^1 \frac{\log x}{\sqrt{1-x}} dx$

(ii) $\int_0^\infty x^{m-1} e^{-x} dx$

Unit - III

[Uniform convergence of sequence and series of functions]

[Marks 16]

6. Answer any ***three*** questions : 2×3

(a) If a sequence of function $\{f_n(x)\}$ be uniformly convergent on $D \subset R$, then prove that the limit function f is bounded on D. 2

(b) If $f_n(x) = x^n$, $x \in [0, 1]$, show that the sequence of functions $\{f_n\}$ is not uniformly convergent on [0, 1]. 2

(c) State Weierstrass M-test for the uniform convergence of a series of function. 2

(d) Find $\underset{x\to 0}{Lt}\sum \frac{\cos nx}{n(n+1)}$. 2

(e) If D be a finite subset of R and a sequence $\{f_n\}$ of real valued functions on D converges pointwise to f, then prove that $\{f_n\}$ converges uniformly to f on D. 2

7. Answer any ***one*** question : 10×1

(a) (i) State and prove Cauchy criterion for the uniform convergence of sequence of functions. 5

(ii) If $\{f_n\}$ be a sequence of function defined on [0, 1] by $f_n(x) = nxe^{-nx^2}$, show that the sequence $\{f_n\}$ is not uniformly convergent on [0, 1]. 5

(b) (i) Let $D \subset R$ and for each $n \in N$, $f_n : D \to R$ is a continuous function on D. If the series $\sum f_n$ be uniformly convergent on D then prove that the sum function S is continuous on D. 4

(ii) Show that the series $\sum \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all real x. If $s(x)$ be the sum function, verify that $s'(x)$ is obtained by term-by-term differentiation. 6

Unit - IV

[Fourier Series]

[Marks 7]

8. Answer any ***one*** question : 2×1

(a) Is $\sum_{1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ is a Fourier Series or not? Justify.

(b) State Dirichlet's conditions for convergence of a Fourier series.

9. Answer any ***one*** question : 5×1

(a) Let $f:[-\pi, \pi] \to \mathbb{R}$ be continuous except for at most a finite number of jumps and is periodic of period 2π then prove that

[Turn Over]

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx$$

where a_n and b_n are the Fourier co-efficients of

$f(x)$ defined by $a_k=\frac{1}{\pi}\int_{\pi}^{\pi}f(t)\cos nt\,dt$, $n\ge 0$

$$=\frac{1}{\pi}\int_{-\pi}^{+\pi}f(t)\sin nt\,dt,$$

for $n\ge 1$.

5

(b) Obtain Fourier series representation of f in $[-\pi,\pi]$ where $f(x)=x\ \forall x\in[-\pi,\pi]$ and hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{\pi}{4}$.

Unit - V

[Power Series]

[Marks 7]

10. Answer any *one* question : 2×1

(a) Let $f(x)$ be the sum of a power series $\Sigma a_n x^n$ on $(-R, R)$ where $R>0$. If $f(x)+f(-x)=0$ $\forall x\in(-R,R)$. Prove that $a_n=0$ for all even positive integer.

(b) Find the interval of convergent of the power series $\sum \frac{(-1)^{n+1}}{n+1}(x+1)^n$.

11. Answer any *one* question : 5×1

(a) Let $\sum a_n x^n$ be a power series with radius of convergence $R\,(> 0)$. Let $f(x)$ be sum of the series on $(-R, R)$ then prove that $f(x)$ is continuous on $(-R, R)$.

(b) Assume the power series

$$\frac{1}{\sqrt{1-x^2}} = 1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+....$$

obtain the power series expansion of $\sin^{-1}x$ and hence deduce

$$1+\frac{1}{2.3}+\frac{1}{2.4.5}+\frac{1.3.5}{2.4.6.7}+...=\frac{\pi}{2}$$