2018

CBCS

3rd Semester

MATHEMATICS

PAPER—C6T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Group Theory—I

Unit—I

1. Answer any *two* questions : 2×2

(a) Is the set R^* of all non-zero real numbers a group with respect to the operations o defined by $a \circ b = |a| b$ for all $a, b \in R^*$? Justify your answer.

(b) Let $(G,*)$ be a group of even order. Show that there exists $a \in G$ show that $a \neq e, a^2 = e$.

(c) Let (G, o) be a group. Define a mapping $f : G \to G$ by $f(x) = x^{-1}, x \in G$. Prove that f is a bijection.

2. Answer any *one* question : 1×5

(a) Show that the set of six transformations f_1, f_2, f_3, f_4, f_5 and f_6 on the set of complex numbers defined by $f_1(z) = z, f_2 = (z) = \frac{1}{z}, f_3(z) = 1 - z; f_4(z) = \frac{z}{z-1}, f_5(z) = \frac{1}{1-z}$ and $f_6(z) = \frac{z-1}{z}$ forms a finite non-Abelian group of order 6 with respect to the composition of mapping.

(b) Construct the dihedral group D_4 from the symmetries of a square. Show that the order of it is 8.

Unit—II

3. Answer any *two* questions : 2×2

(a) A non-Abelian group have an Abelian subgroup. Justify the statement with example.

(b) In a group $(G, \cdot), (ab)^3 = a^3 b^3 \forall a, b \in G$. Show that $H = \left\{x^3 : x \in G\right\}$ is a subgroup of G.

(c) Let (G, o) be a group and H, K are subgroups of (G, o). Then show that $H \cap K$ is a subgroup of (G, o)

4. Answer any *two* questions : 2×5

(a) Prove that H is a subgroup of Z_{12}

where $H = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$.

(b) Let H, K be subgroups of a group G. Prove that set HK is a subgroup of G iff $HK = KH$.

where $HK = \left\{hK : h \in H \text{ and } k \in K\right\}$
$kH = \left\{Kh : k \in K \text{ and } h \in H\right\}$

(c) Let H be a subgroup of a group G and $a \in G$. Define normalizer of a in G and centralizer of H in G. Show that centralizer of H and normalizer of H in G are not same. Justify your answer with example.

Unit—III

5. Answer any *two* questions : 2×2

(a) Let G be a finite group, A and B be two subgroups of G such that $A \subseteq B \subseteq G$. Prove that,

$$[G:A]=[G:B][B:A]$$

(b) Show that a cyclic group with only one generator can have at most two elements.

(c) Determine all distinct left cosets of A_3 in S_3.

6. Answer any *one* question : 1×10

(a) (i) Let H be a subgroup of a group G. Then show that the set of all distinct left cosets of H in G have the same cardinality.

(ii) Show that the number of even permutation of a finite set (containing at least two elements) is equal to the number of odd permutation on it. 5+5

(b) Prove that, a finite group of order n is cyclic if and only if it has an element of order n. Also prove that every subgroup of a cyclic group is cyclic. 5+5

Unit—IV

7. Answer any *two* questions : 2×2

(a) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \text{ are real and } ac \neq 0 \right\}$ be a group under matrix multiplication. Show that $N = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \text{ is a real number} \right\}$ is a normal subgroup of G.

(b) If H be a subgroup of a commutative group G then show that G/H is commutative.

(c) Show that if p is a prime number, then any group G of order $2p$ has a normal subgroup of order p.

8. Answer any *one* question : 1×10

(a) Define centre $Z(G)$ of a group G. Prove that $Z(G)$ is a normal subgroup of $(G, \circ)$. Also prove that $mn = nm$ $\forall m \in M$ and $n \in N$, where M and N are two normal subgroups of a group G. Show that $M \cap N = \{e\}$, e being the identity element in G. 2+4+4

(b) State and prove Cauchy's theorem for finite Abelian groups. 2+8

Unit—V

9. Answer any *two* questions : 2×2

(a) Define $f : (S_3, \circ) \to (\{1,-1\}, \bullet)$ by

$$f(\alpha) = 1, \text{ if } \alpha \text{ is an even permutation in } S_3$$
$$= -1, \text{ if } \alpha \text{ is an odd permutation in } S_3.$$

Show that f is homomorphism from $(S_3, \circ)$ to $(\{1,-1\}, \bullet)$, o is the composition of mapping.

(b) Show that Ker ϕ (Kernel of homomorphism ϕ) from (G, o) to $(G, *)$ is a normal subgroup of G.

(c) Let $GL(2, R)$ be the group of non-singular real matrices under multiplication, R^* be the group of non-zero reals under multiplication and a function

$F : GL(2, R) \to R^*$ is defined by $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

Show that f is a homomorphism.

10. Answer any *one* question : 1×5

(a) Prove that every finite group G is isomorphic to a permutation group. 5

(b) If H and K are two normal subgroup of G such that $H \subseteq K$, then show that $\frac{G}{K} \cong \frac{G/H}{K/H}$. 5