2018

CBCS

3rd Semester

MATHEMATICS

PAPER—C5T

(Honours)

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Theory of Real Functions
and
Introduction to Metric Space

UNIT—1

1. Answer any *Three* quesitons : 3×2

(a) Prove that $\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right) = 0$ 2

(Turn Over)

(b) Let $D \subset R$ and $f : D \to R$ be a function. If c be an isolated point of D then prove that f is continuous at c. 2

(c) State the sequential criterion for the continuity of a function f at a point c. 2

(d) By Cauchy's principle prove that $\lim_{x \to 0} \cos\frac{1}{x}$ does not exist. 2

(e) Show that $f(x) = x^2, x \in R$ is not uniformly continuous on R. 2

2. Answer any *one* question : 1×5

(a) Let $I = [a, b]$ be a closed and bounded interval and $f : [a, b] \to R$ be continuous on I. Then prove that $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval.

(b) Let $f : [a, b] \to R$ and $g : [a, b] \to R$ be Continuous on $[a, b]$ and let $[f(a) - g(a)].[f(b) - g(b)] < 0$. Show that the exists a point c in (a, b) such that $f(c) = g(c)$.

Deduce that $\cos x = x^2$ for some $x \in \left(0, \frac{\pi}{2}\right)$ 3+2

3. Answer any *one* question : 1×10

(a) i) Let the functions $f: R \to R$ and $g: R \to R$ are both continuous on R. Then prove that the set $S = \{x \in R : f(x) = g(x)\}$ is a closed set in R. 4

ii) Explain for continuity the function f defined by $f(x) = \lim_{n\to\infty} \dfrac{e^x - x^n \sin nx}{1+x^n}$ $\left(0 \le x \le \frac{\pi}{2}\right)$ at $x = 1$. Explain why the function f does not vanish any where in $\left[0, \frac{\pi}{2}\right]$ although $f(0)f\left(\frac{\pi}{\sqrt{2}}\right) < 0$.

(b) i) Let $[a, b]$ be a closed and bounded interval and $f: [a, b] \to R$ be continuous on $[a, b]$. If $f(a).f(b)<0$ then prove that $f(x) = 0$ has at least one root in (a, b). Hence show that any algebraic equation of an odd power with real co-efficients has at least one real root. 5+2

ii) Show that if a function $f: [a, b] \to R$ is uniformly continuous on (a, b) then it is continuous on (a, b). Is the converse true ? Justify. 2+1

UNIT—2

4. Answer any *two* quesitons : 2×2

(a) Let I be an interval and $c \in I$. Let the function $f : I \to R$ is differentiable at c. Then prove that if $k \in R$, kf is differentiable at c and $(kf)'(c) = kf'(c)$.

(b) Prove that $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1,\ x > 0$.

(c) Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in $(0, 1)$.

5. Answer any *two* questions : 2×5

(a) Let $I = [a,b]$ and $f : I \to R$ be differentiasle on I. If $f'(a) \cdot f'(b) < 0$ then prove that there exists a point $c \in (a,b)$ s.t. $f'(c) = 0$. 5

(b) State Cauchy mean value theorem and deduce Lagrange mean value theorem from it. Give geometrical interpretation of Lagrange mean value theorem. 1+2+2

(c) Let $f(x)=e^{-\frac{1}{x^2}}\sin\left(\frac{1}{x}\right)$ when $x \neq 0$ and $f(0) = 0$

Show that at every point f has a differential coefficient and this is continuous at $x = 0$. 5

UNIT—3

6. Answer any *two* questions : 2×2

(a) Use Taylor's theorem to prove that $\cos x \geq 1-\frac{x^2}{2}$, for $-\pi < x < \pi$.

(b) Examine if f has a local maximum or a local minimum at 0 where $f(x) = x - [x]$.

(c) Find θ, if $f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x+\theta h)$ $0<\theta<1$ and $f(x) = x^3$.

7. Answer any *one* question : 1×10

(a) i) State and prove Taylor's theorem with Langrange form of remainder. 2+4

(ii) If $f(x) = \sin x$ prove that $\lim_{h \to 0} \theta = \frac{1}{\sqrt{3}}$ where q is given by $f(h) = f(0) + hf'(\theta h)$, $0 < q < 1$ 4.

(b) (i) State and prove Maclaurin's infinite series of a function f. 5

(ii) Derive infinite series expansion of the function $\log(1 + x)$, $x > -1$. 5

UNIT—4

8. Answer any *three* questions : 3×2

(a) Define separable metric space with example.

(b) Let (X, d) be a metric space. Prove that a non empty open subset G can be expressed as a union of open balls.

(c) Let $X = R^2$, the set of all points in the co-ordinate plane. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X define $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Show that $d(x, y)$ is a metric space.

9. Answer any *one* question : 1×5

(a) Define closer of a set S in a metric space. Prove that in any metric space closer of a set S is a closed set.

(b) Let X be the set of all real valued continuous functions defined on the closed interval $[a, b]$. If for $x, y \in X$, we define $d(x,y) = \underset{a \le t \le b}{\text{Sup}} |x(t) - y(t)|$. Then prove that (X, d) is a metric space. 5

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